

The von Neumann algebra generated by t -gaussians

Éric Ricard

Abstract

We study the t -deformation of gaussian von Neumann algebras. When the number of generators is fixed, it is proved that if t sufficiently close to 1, then these algebras do not depend on t . In the same way, the notion of conditionally free von Neumann algebras often coincides with freeness.

1 Introduction

The notions of free and non commutative probabilities originally appeared in the works of Voiculescu in the 80's (see [14] for instance) to study von Neumann algebras, in particular the von Neumann algebra $L(\mathbb{F}_n)$ associated to the free group with n generators. Since then, this domain has expanded rapidly, and is now considered as a subject in itself. It has connexions with combinatorics, classical probabilities, mathematical physics and of course operator algebras. Naturally, people become interested in finding generalizations or deformations of the free probability or the free product constructions to obtain new non commutative probability spaces, especially from the combinatoric point of view. One of the first successful attempt was made by Bożejko and Speicher ([5], see also [3]) in introducing the so called q -deformation of the free factor. These von Neumann algebras were studied in the last years, at the moment it is known that they share some properties with the free group algebras : they are factor [12], non injective [11], and solid [13] (for some values of the parameter q). Nevertheless, it is still unknown if they really differ from the free group algebras.

With the same spirit, Bożejko, Leinert and Speicher introduced in [4] the concept of conditional freeness. They were able to describe the combinatoric underlying this notion in a way similar to that of the freeness in terms of Cauchy Transforms and their reciprocal. The school of Accardi also developed its own deformation by considering the algebra generated by position operators on an interacting Fock space (see [1]). Unfortunately, very little work on these two topics were concerned with the study of the resulting von Neumann object. One of the motivation for the present work is to start such a study like in the q -case. We will mainly focus on specific examples of von Neumann algebras, the t -gaussian von Neumann algebras. They have the advantage to be both a model for conditional freeness and an algebra on an interacting Fock space. They even appear as a limit object for the theory of the t -convolution of measures of Bożejko and Wysoczański in [6]. So they seem to be quite central in all those deformations of the free probability.

As an illustration, we recall the definition of the conditional freeness, and explain how the t -gaussian algebra appears. Let (A_i, ϕ_i, ψ_i) be $*$ -algebras equipped with a pair of states. Assume that all A_i 's lie in a bigger algebra A with a state ϕ . Then we say that the algebras A_i 's are conditionally free for ϕ provided that whenever

$$a_j \in A_{i_j}, i_1 \neq i_2 \neq \dots \neq i_n, \psi_{i_j}(a_j) = 0$$

we have

$$\phi(a_{i_1} \dots a_{i_n}) = \phi_{i_1}(a_{i_1}) \dots \phi_{i_n}(a_{i_n})$$

It is clear that for any given (A_i, ϕ_i, ψ_i) it is possible to construct ϕ on the free product $A = *_i A_i$ so that the A_i 's are conditionally free. In the situation where $\phi_i = \psi_i$, one recovers the classical

notion of freeness, that is ϕ is the free product of the ψ_i . We let ψ be the free product of the ψ_i on A .

Given some probability measures μ_i, ν_i (with compact support), one can consider them as states on the space of polynomials in one variable $\mathbb{C}[X_i]$ in a natural manner by the formula

$$\mu_i(X_i^k) = \int x^k d\mu_i(x).$$

The free product of n algebras of polynomials in one variable is exactly the space of polynomials in n non commuting variables $\mathbb{C}\langle X_i \rangle_{i \leq n}$. So, using the above construction, from the measures μ_i, ν_i one can build two new states ϕ and ψ on $\mathbb{C}\langle X_i \rangle$, so that the algebras $(\mathbb{C}[X_i], \mu_i, \nu_i)$ are conditionally free in $(\mathbb{C}\langle X_i \rangle, \phi, \psi)$. The distribution of $X_1 + X_2$ with respect to ϕ is the compactly supported measure μ so that

$$\phi((X_1 + X_2)^k) = \int x^k d\mu(x).$$

The c-free convolution of (μ_1, ν_1) and (μ_2, ν_2) is then defined as

$$(\mu_1, \nu_1) \boxplus_c (\mu_2, \nu_2) = (\mu, \nu_1 \boxplus \nu_2)$$

where \boxplus means the usual free convolution (i.e. $\nu_1 \boxplus \nu_2$ is the distribution of $X_1 + X_2$ with respect to ψ). This operation \boxplus_c on couples of measures is commutative and associative. Limits theorems were obtained in this framework. For instance, a central limit theorem consists in finding the possible limit distributions of $X_1 + \dots + X_n/\sqrt{n}$, where the X_i are identically distributed (and centered) and are conditionally free. The measures (μ, ν) appearing at the limit, are parameterised by the second moments of μ and ν . So up to normalization $\mu_t(X^2) = 1$ and $\nu_t(X^2) = t$. The von Neumann algebra arising from the GNS construction of the first state on the conditional free product $*_{i=1}^n (\mathbb{C}[X_i], \mu_t, \nu_t)$, is $\Gamma_{t,n}$ the t -gaussian algebra with n generators. When $t = 1$, this is just the classical objects of free probabilities, that is $\Gamma_{1,n} = L(\mathbb{F}_n)$.

The main result of this paper states that for $n \geq 2$

$$\Gamma_{t,n} = \begin{cases} \Gamma_{1,n} & \text{if } t \in \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}} \right] \\ \Gamma_{1,n} \oplus \mathbb{B}(\ell_2) & \text{otherwise} \end{cases},$$

where as usual $\mathbb{B}(\ell_2)$ stands for the bounded operators on a separable Hilbert space. Consequently, this result is a bit disappointing for operator algebras, as the deformations may not produce new objects.

In the next section, we give a precise construction of $\Gamma_{t,n}$ arising from the theory of one mode interacting Fock space and very basic results about it. The third section is devoted to the proof of the main result. In the last section, we give some conditions that ensure the equality between the reduced free product of commutative von Neumann algebras (A_i, ψ_i) and their conditional reduced free product (A_i, ϕ_i, ψ_i) . These conditions are satisfied for the t -gaussian algebras (in the first case), however, we prefer to give slightly different proofs for these examples as one can extract information from them to get some partial results on interacting Fock space.

2 Basics

In the whole paper, t will be a positive real number $t > 0$. We will also use standard notations, $\mathbb{B}(H)$ and $\mathbb{K}(H)$ will denote the bounded and the compact operators on the Hilbert space H .

For a given Hilbert space \mathcal{H} , with real part $\mathcal{H}_{\mathbb{R}}$, equipped with a scalar product $\langle \cdot, \cdot \rangle$, we denote by \mathcal{F}_1 , the Fock space built on \mathcal{H} :

$$\mathcal{F}_1 = \mathbb{C}\Omega \oplus_{k \geq 1} \mathcal{H}^{\otimes k}$$

More generally, the t -deformed Fock space is given by

$$\mathcal{F}_t = \mathbb{C}\Omega \oplus_{k \geq 1} t^{k-1} \mathcal{H}^{\otimes k}$$

Here, $t^{k-1}\mathcal{H}^{\otimes k}$ simply means the space $H^{\otimes k}$ where the scalar product is multiplied by t^{k-1} .

In the most part of the paper, we will assume that \mathcal{H} is finite dimensional, say $\dim \mathcal{H} = n$. Let $(e_i)_{i=1\dots n}$ be a real basis of $\mathcal{H}_{\mathbb{R}}$. Then one can define a canonical basis for \mathcal{F}_t . Its elements $e_{\underline{i}}$ are indexed by words \underline{i} in the n letters $1, \dots, n$, and

$$e_{\underline{i}} = \frac{1}{\sqrt{t}^{k-1}} e_{i_1} \otimes \dots \otimes e_{i_k} \quad \text{for } \underline{i} = i_1 \dots i_k$$

As usual, for $e \in \mathcal{H}_{\mathbb{R}}$, the creation operator associated to e is defined on \mathcal{F}_t by

$$\begin{aligned} l_t(e)\Omega &= e \\ l_t(e)(h_1 \otimes \dots \otimes h_k) &= e \otimes h_1 \otimes \dots \otimes h_k \end{aligned}$$

It is well known that $l_t(e)$ extends to a bounded operator on \mathcal{F}_t . Its adjoint is given by

$$\begin{aligned} l_t(e)^*\Omega &= 0 \\ l_t(e)^*h &= \langle h, e \rangle \Omega \\ l_t(e)^*(h_1 \otimes \dots \otimes h_k) &= t \langle h_1, e \rangle h_2 \otimes \dots \otimes h_k \end{aligned}$$

The t -gaussian associated to e is the operator $s^t(e) = l_t(e) + l_t(e)^*$.

We are interested in the von Neumann generated by the t -gaussians. Let $s_i^t = l_t(e_i) + l_t(e_i)^*$ for $i = 1, \dots, n$. $C_{t,n}$ and $\Gamma_{t,n}$ refer to the C^* -algebra and the von Neumann algebras generated by the s_i^t 's, $i = 1, \dots, n$:

$$\Gamma_{t,n} = \{l_t(e) + l_t(e)^* ; e \in \mathcal{H}_{\mathbb{R}}\}'' \subset \mathbb{B}(\mathcal{F}_t)$$

This deformation of the Fock Hilbert space is a particular case of interactive Fock spaces. This type of objects was introduced in [1]. The basic idea is to modify the scalar product at each level $\mathcal{H}^{\otimes k}$ by a positive scalar λ_k .

The vector state $\langle \cdot, \Omega \rangle$ on $\mathbb{B}(\mathcal{F}_t)$ is called the vacuum state and will be denoted by ϕ . In the case $t = 1$, ϕ is a trace on $\Gamma_{1,n}$, so we will prefer the notation τ .

In the following, we sum up all basic results about the von Neumann algebra generated by a single t -gaussian s^t when $\mathcal{H} = \mathbb{C}$.

Proposition 2.1 $\Gamma_{t,1}$ is in GNS position with respect to ϕ , which is faithful on it.
The distribution of s^t with respect to ϕ is given by

$$\begin{aligned} \frac{1}{2\pi} \frac{\sqrt{4t-x^2}}{1-(1-t)x^2} \mathbb{1}_{[-2\sqrt{t}, 2\sqrt{t}]} dx &\quad \text{if } t \geq \frac{1}{2} \\ \frac{1}{2\pi} \frac{\sqrt{4t-x^2}}{1-(1-t)x^2} \mathbb{1}_{[-2\sqrt{t}, 2\sqrt{t}]} dx + \frac{1-2t}{2-2t} (\delta_{\frac{1}{-\sqrt{1-t}}} + \delta_{\frac{1}{\sqrt{1-t}}}) &\quad \text{if } t < \frac{1}{2} \end{aligned}$$

The map $\rho : \Gamma_{t,1} \rightarrow \Gamma_{1,1}$ given by $\rho(s^t) = \sqrt{t}s^1$ extends to a normal representation.

Moreover, given $i \leq n$, there are $*$ -isometric normal representations $\pi_i : \Gamma_{t,1} \rightarrow \Gamma_{t,n}$ given by $\pi(s^t) = s_i^t$.

Proof : The computations of the distribution of s^t can be found in [4, 6, 15]. The G -transform of the distribution of s^t for ϕ is given by

$$G_{s^t}(z) = \frac{(\frac{1}{2}-t)z + \frac{1}{2}\sqrt{z^2-4t}}{z^2(1-t)-1} = \cfrac{1}{z-\cfrac{1}{z-\cfrac{t}{z-\cfrac{t}{z-\cfrac{t}{\ddots}}}}}$$

It is obvious that Ω is cyclic for $\Gamma_{t,1}$ so is also separating since the algebra is commutative and ϕ is faithful. In particular, the spectral measure of $\sqrt{t}s^1$ is absolutely continuous with respect to that of s^t , this implies that the map $\rho : \Gamma_{t,1} \rightarrow \Gamma_{1,1}$ is well defined and normal.

Note that the matrix of s^t in the natural orthonormal basis is given by

$$\begin{bmatrix} 0 & 1 & & & & \\ 1 & 0 & \sqrt{t} & & & \\ & \sqrt{t} & 0 & \sqrt{t} & & \\ & & \ddots & \ddots & \ddots & \\ & & & \sqrt{t} & 0 & \sqrt{t} \\ & & & & \ddots & \ddots & \ddots \end{bmatrix}$$

So at the C^* -level, ρ is just the restriction of the Calkin map ($\rho : \mathbb{B}(\mathcal{F}_1) \rightarrow \mathbb{B}(\mathcal{F}_1)/\mathbb{K}(\mathcal{F}_1)$, where \mathbb{K} stands for the compact operators) since $s^t - \sqrt{t}s^1$ is of rank 2 and $\Gamma_{1,1}$ does not meet the compact operators (no atoms in the measure).

In $\Gamma_{t,n}$, s_i^t is unitarily equivalent to $s^t \oplus (\sqrt{t}s^1)_\infty$ corresponding to its decomposition in reducing subspaces (they are indexed by reduced word starting by a letter which is not a i). \blacksquare

Remark 2.2 There are natural isometries between all \mathcal{F}_t by identifying the canonical basis. This allows to consider all $\Gamma_{t,n}$ as subalgebras of $\mathbb{B}(\mathcal{F}_1)$. When we will talk about the Calkin map, we mean the classical quotient map $\rho : \mathbb{B}(\mathcal{F}_1) \rightarrow \mathbb{B}(\mathcal{F}_1)/\mathbb{K}(\mathcal{F}_1)$. Using this identification, we have $\rho(C_{t,n}) = \rho(C_{1,n})$ for any $0 < t$ and $1 \leq n < \infty$.

Remark 2.3 From the densities, it is clear that as von Neumann algebras, we have

$$\Gamma_{t,1} = \begin{cases} \Gamma_{1,1} \oplus \mathbb{C}^2 & \text{if } t < \frac{1}{2} \\ \Gamma_{1,1} & \text{if } t \geq \frac{1}{2} \end{cases}$$

Let $A_i \approx \Gamma_{t,1}$ be the algebra generated by s_i^t in $\Gamma_{i,n}$. By the previous proposition, there are two given states on A_i . One is coming from the vacuum state (denoted by ϕ , no confusion since it coincides with the vacuum on $\Gamma_{t,n}$) and another one coming from the vacuum state on $\Gamma_{1,1}$ that is $\tau\rho\pi_i^{-1}$ (call it ψ even if it depends on i).

The following is well known

Proposition 2.4 *The algebras (A_i, ϕ, ψ) are conditionally free with respect to the vacuum state, that is*

$$\phi(a_1 \dots a_p) = \phi(a_1) \dots \phi(a_p) \quad \text{whenever } a_j \in A_{i_j}, i_1 \neq i_2 \neq \dots \neq i_p \text{ and } \psi(a_j) = 0.$$

Moreover $(\Gamma_{t,n}, \phi, \Omega)$ is in GNS position.

We postpone the proof of this fact to the next section.

Let U be an orthogonal transformation on $\mathcal{H}_{\mathbb{R}}$ and still denote its tensorization with $\text{Id}_{\mathbb{C}}$ by U . The first quantization $\Gamma(U)$ of U is the unitary on \mathcal{F}_t given by

$$\Gamma(U) = \text{Id}_{\mathbb{C}\Omega} \oplus_{k \geq 1} U^{\otimes k}$$

Then, $\Gamma_{t,n}$ is stable by conjugation by $\Gamma(U)$ and for any $e \in \mathcal{H}_{\mathbb{R}}$,

$$\Gamma(U)s^t(e)\Gamma(U)^* = s^t(Ue)$$

3 The von Neumann algebra $\Gamma_{t,n}$

3.1 Factoriality

The following is well known to specialists :

Proposition 3.1 *Let $M \subset \mathbb{B}(H)$ be a von Neumann algebra, if M contains a non zero compact operator, then either $M = \mathbb{B}(K) \otimes \text{Id}_d$ with $H = K^d$ and d is the smallest rank of a non zero compact projection in M or M is not a factor and has a direct summand isomorphic to $\mathbb{B}(K)$ for some K .*

Proof : We know that M contains a finite rank projection. So let p be a finite rank projection of minimum rank $d \geq 1$. Let $(p_i)_{i \in I}$ be a maximal family of mutually orthogonal projections equivalent to p (so all p_i 's have the same finite rank). Let $q = \sum p_i$, we show that q is central.

If q is not central then there is some $x \in M$ so that $(1 - q)xq \neq 0$. Hence we can assume that $\text{Ran } x$ is orthogonal to $\text{Ran } q$ and $xq \neq 0$. There must be an i so that $xp_i \neq 0$. Consider $xp_i x^*$, it is non zero and of rank less or equal to d . By definition of d , $p_0 = xp_i x^*$ must have at most one non zero eigenvalue, and has at least one as $p_0 \neq 0$. So p_0 is a multiple (say 1) of a projection of rank d . Moreover, $p_i x^* x p_i$ is also a non zero projection hence it must be equal to p_i . So p_0 is equivalent to p . But the range of p_0 is orthogonal to q so $p_0 q = q p_0 = 0$, this contradicts the maximality of I . So q is central.

Let $u_{i,j}$ be the partial isometries between p_i and p_j $i, j \in I$. It is not hard to see that $u_{i,j}$ is a system of matrix units that generates qM . So if $q = 1$ then M is a factor, else M is not a factor and has a direct summand isomorphic to $\mathbb{B}(\ell_2(I))$. \blacksquare

In the case $d = 1$, the argument is much simpler : let ξ be in the range of p as above. Then M contains all projections on $K = \overline{M \cdot \xi}$ (using conjugation of p). So if $K \neq H$, then the projection q onto K belongs to both M' (by definition) and M (because $q = \sum p_i$ with p_i projection onto lines corresponding to an onb in K), so M is not a factor and has a direct summand isomorphic to $\mathbb{B}(K)$. If $K = H$ the same kind of arguments gives that $M = \mathbb{B}(K)$.

3.2 Orthogonal polynomials

Let U_n be the Tchebychef polynomial of the second kind of degree n .

Proposition 3.2 *The orthonormal polynomials for s^t with respect to ϕ are given by :*

$$\begin{aligned} v_0(X) &= 1 & v_1(X) &= X = \sqrt{t} U_1\left(\frac{X}{2\sqrt{t}}\right) \\ v_n(X) &= \sqrt{t} \left(U_n\left(\frac{X}{2\sqrt{t}}\right) - \left(\frac{1}{t} - 1\right) U_{n-2}\left(\frac{X}{2\sqrt{t}}\right) \right) & \text{for } n \geq 2 \end{aligned}$$

The orthonormal polynomials for $\sqrt{t}s^t$ are given by $u_n(X) = U_n\left(\frac{X}{2\sqrt{t}}\right)$.

Proof : According to the continued fraction decomposition of the G -transform of the measure, the unital orthogonal polynomials must satisfy the relations

$$XP_n(X) = P_{n+1}(X) + tP_{n-1}(X) \quad \text{for } n \geq 2$$

$$XP_1(X) = P_2(X) + P_0(X), \quad P_0(X) = 1, \quad P_1(X) = X$$

Then one deduces that this relation is satisfied with $P_n(X) = 2^n \sqrt{t}^{n-1} v_n(X)$. See also [2, 15] for detailed proofs. \blacksquare

The polynomials u_n and v_n are related one to each other, letting $\alpha = \frac{1}{t} - 1$

$$\begin{aligned} v_n &= \sqrt{t} (u_n - \alpha u_{n-2}) \\ (R) \quad u_{2n} &= \alpha^n v_0 + \frac{1}{\sqrt{t}} \sum_{k=1}^n \alpha^{n-k} v_{2k} \\ u_{2n+1} &= \frac{1}{\sqrt{t}} \sum_{k=0}^n \alpha^{n-k} v_{2k+1} \end{aligned}$$

Lemma 3.3 *At the algebraic level, we have for $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$, with $\underline{i} = i_1^{\alpha_1} \dots i_l^{\alpha_l}$:*

$$u_{\alpha_1}(s_{i_1}^t) \dots u_{\alpha_{l-1}}(s_{i_{l-1}}^t) v_{\alpha_l}(s_{i_l}^t) \Omega = e_{\underline{i}}$$

Proof : We have $v_{\alpha_l}(s_{i_l}^t)\Omega = e_{i_l}^{\alpha_l}$ as the v_j are the orthonormal polynomials for $s_{i_l}^t$. Now, $s_{i_{l-1}}^t$ acts on tensors starting with a letter i_l exactly as $\sqrt{t}s_{i_{l-1}}^1$ so $u_{\alpha_{l-1}}(s_{i_{l-1}}^t)v_{\alpha_l}(s_{i_l}^t)\Omega = e_{i_{l-1}^{\alpha_{l-1}}i_l^{\alpha_l}}^t$, and so on. \blacksquare

Proof of Proposition 2.4 : Using multilinearity, we need to show that for $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$, we have

$$\phi(u_{\alpha_1}(s_{i_1}^t) \dots u_{\alpha_l}(s_{i_l}^t)) = \phi(u_{\alpha_1}(s_{i_1}^t)) \dots \phi(u_{\alpha_l}(s_{i_l}^t))$$

From the relations (R), we have

$$\phi(u_{\alpha_k}(s_{i_k}^t)) = \begin{cases} \alpha_k^{\alpha_k/2} & \text{if } \alpha_k \text{ is even} \\ 0 & \text{otherwise} \end{cases}$$

But by the previous lemma

$$0 = \frac{1}{\sqrt{t}}\phi(u_{\alpha_1}(s_{i_1}^t) \dots v_{\alpha_l}(s_{i_l}^t)) = \phi(u_{\alpha_1}(s_{i_1}^t) \dots u_{\alpha_l}(s_{i_l}^t)) - \alpha\phi(u_{\alpha_1}(s_{i_1}^t) \dots u_{\alpha_{l-2}}(s_{i_l}^t))$$

Then we conclude by induction on $\sum \alpha_k$. \blacksquare

The statement about the GNS position can then be easily deduced from the conditional freeness (see Lemma 4.1). \blacksquare

Let

$$\eta = \Omega - \sqrt{t}\alpha \sum_{k=1}^n e_{kk}.$$

Define a functional on $\mathbb{B}(\mathcal{F}_t)$ by :

$$\tilde{\psi}(x) = \langle x\Omega, \eta \rangle$$

Lemma 3.4 *The functional $\tilde{\psi}$ coincides with $\tau\rho$ on $C_{t,n}$.*

Proof : The set of polynomials in the letters s_i^t is dense in $C_{t,n}$. As a consequence, the linear span of products $u_{d_1}(s_{i_1}^t) \dots u_{d_m}(s_{i_m}^t)$, with $d_j \geq 1$, $m \geq 0$ and $i_1 \neq i_2 \neq \dots \neq i_m$ is dense in $C_{t,n}$. So we only need to prove that as soon as $m \geq 1$ (using freeness) :

$$\begin{aligned} 0 = \langle u_{d_1}(s_{i_1}^t) \dots u_{d_m}(s_{i_m}^t)\Omega, \eta \rangle &= \langle u_{d_1}(s_{i_1}^t) \dots u_{d_m}(s_{i_m}^t)\Omega, \left(1 - \sqrt{t}\alpha \sum_{k=1}^n v_2(s_k^t)\right)\Omega \rangle \\ &= \phi\left(\left(1 - \sqrt{t}\alpha \sum_{k=1}^n v_2(s_k^t)\right)u_{d_1}(s_{i_1}^t) \dots u_{d_m}(s_{i_m}^t)\right) \end{aligned}$$

So we need to evaluate expressions of the form

$$\phi(ab_1 \dots b_l)$$

with $a \in A_{j_0}$, $b_u \in A_{j_u}$ with $j_0 \neq j_1 \neq \dots \neq j_{l+1}$ and $\psi(b_i) = 0$ (with possibly $l = 0$). Using conditional freeness, and denoting by $b = b_1 \dots b_l$ and $x = \phi(b) = \phi(b_1) \dots \phi(b_l)$

$$\begin{aligned} \phi(ab) &= \phi((a - \psi(a))b) + \psi(a)\phi(b) \\ &= \phi(a)\phi(b) \end{aligned}$$

Using this equality, we get as $\phi(v_2(s_i^t)) = 0$

$$\langle u_{d_1}(s_{i_1}^t) \dots u_{d_m}(s_{i_m}^t)\Omega, \eta \rangle = \phi\left(\left(1 - \sqrt{t}\alpha v_2(s_{i_1}^t)\right)u_{d_1}(s_{i_1}^t) \dots u_{d_m}(s_{i_m}^t)\right)$$

In this scalar product, there will be a factor

$$\phi\left(\left(1 - \sqrt{t}\alpha v_2(s_{i_1}^t)\right)u_{d_1}(s_{i_1}^t)\right)$$

From the formula above, if d_1 is odd then u_{d_1} is in the span of $\{v_{2k+1} ; k \geq 0\}$, so this quantity is 0. If $d_1 = 2p \geq 2$ then

$$\begin{aligned} \phi\left(\left(1 - \sqrt{t}\alpha v_2(s_{i_1}^t)\right)u_{d_1}(s_{i_1}^t)\right) &= \phi\left(\left(1 - \sqrt{t}\alpha v_2(s_{i_1}^t)\right)\left(\alpha^p + \frac{1}{\sqrt{t}} \sum_{j=1}^p \alpha^{p-j} v_{2j}(s_{i_1}^t)\right)\right) \\ &= \alpha^p - \alpha \cdot \alpha^{p-1} = 0 \end{aligned}$$

■

Corollary 3.5 *The map ρ extends to a normal (surjective) representation $\Gamma_{t,n} \rightarrow \Gamma_{1,n}$.*

Proof : From the above lemma, it follows that $\tilde{\psi}$ is positive on $C_{t,n}$. As it is normal, it extends to a normal state on $\Gamma_{t,n}$ (that we will denote by ψ). Now, the GNS representation of ψ gives the normal representation. ■

Corollary 3.6 $\Gamma_{t,n}$ has a direct summand isomorphic to $\Gamma_{1,n}$.

Remark 3.7 If the map ρ is not isomorphic, then $\Gamma_{t,n}$ is not a factor. Otherwise it is a type II_1 factor.

Remark 3.8 Denoting by $c^t = (s_1^t)^2 + \dots + (s_n^t)^2$, we have

$$\eta = \alpha\left(\left(n + \frac{1}{\alpha}\right) - c^t\right)\Omega$$

We will study this operator in the next sections.

3.3 Case $t < n/(n + \sqrt{n})$ and $t > n/(n - \sqrt{n})$

3.3.1 Case $t < 1/2$

In this section, we focus on the case $t < 1/2$. It is a particular case of the next one but the arguments are simpler.

When $t < 1/2$, the measure of s^t contains one atom at the point $\frac{1}{\sqrt{1-t}} > \sqrt{2t}$. Since in the Calkin algebra $\rho(s^t) = \rho(\sqrt{t}s^1) \approx \sqrt{t}s^1$, we deduce that $C_{t,1}$ contains a compact operator, corresponding to the point projection on $\frac{1}{\sqrt{1-t}}$ (denoted by P) computed on s^t . Since Ω is cyclic, this projection is one dimensional. From the decomposition $s_i^t \approx s^t \oplus (\sqrt{t}s)_\infty$, we also get that $p = P(s_i^t)$ is a one dimensional projection. The range of p is the linear span of

$$\xi_i = \Omega + \frac{1}{\sqrt{1-t}} \sum_{k=1}^{\infty} \alpha^{\frac{1-k}{2}} e_{i^k}$$

This vector makes sense as $0 < \alpha < 1$ when $0 < t < 1/2$.

According to the Remark 3.7 and Proposition 3.1, on one hand $\Gamma_{t,n}$ ($\infty > n \geq 2$) is either $\mathbb{B}(\ell_2(I))$ or not a factor. And on the other hand it is not a factor or is type II_1 .

Corollary 3.9 *For $t < 1/2$, the von Neumann algebra $\Gamma_{t,n}$ is not a factor for $2 \leq n < \infty$. Moreover it has a direct summand isomorphic to $\mathbb{B}(\ell_2)$ and another one to $\Gamma_{1,n}$ and the state ϕ is not faithful.*

Lemma 3.10 *The vector ξ_1 is not cyclic for $\Gamma_{t,n}$.*

Proof : It suffices to show that η is orthogonal to $\Gamma_{t,n}\xi_1$. But $\xi_1 = P(s_1^t)\Omega$, for P a multiple of the Dirac function at $\frac{1}{\sqrt{1-t}}$. For any $x \in \Gamma_{t,n}$, we have

$$\langle x\xi_1, \eta \rangle = \langle xP(s_1^t)\Omega, \eta \rangle = \tau(\rho(xP(s_1^t)))$$

But we know that $\rho(P(s_1^t)) = 0$ since $P(s_1^t)$ is compact. ■

Proof : The projection $P(s_1^t)$ is of course minimal, so as $\Gamma_{t,n}\xi_1$ is infinite dimensional (for $n \geq 2$), for the $\mathbb{B}(\ell_2(I))$ summand provided by Proposition 3.1, I is infinite. Then, the state ϕ can not be faithful because of the $\mathbb{B}(\ell_2)$ summand. ■

3.3.2 Case $t < n/(n + \sqrt{n})$ and $t > n/(n - \sqrt{n})$

Lemma 3.11 *For $t \notin \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}}\right]$, the C^* -algebra $C_{t,n}$ contains a compact operator.*

First proof : It is possible to compute explicitly the distribution of $c^t = (s_1^t)^2 + \dots + (s_n^t)^2$ for ϕ using the R -transforms machinery developed in [4]. We denote by $\gamma_i = s_i^{t^2}$, these variables are conditionally free with respect to the distribution of ts^{1^2} . From Theorem 5.2 in [4], we know the following relations

$$R_{c^t}(z) = nR_{\gamma_i}(z)$$

$$G_{c^t}(z) = \frac{1}{z - R_{c^t}(G_{tc^1}(z))}$$

The R and G -transforms of tc^1 are obtained as usual using freeness and a change of variable. The computation gives

$$G_{\gamma_i}(z) = \frac{1}{2} \frac{1 - 2t + \sqrt{1 - 4t/z}}{z(1-t) - 1}$$

$$R_{\gamma_i}(z) = \frac{1}{1 - tz}$$

$$R_{c^t}(z) = \frac{n}{1 - tz}$$

$$R_{tc^1}(z) = \frac{nt}{1 - tz}$$

$$G_{tc^1} = \frac{(1-n)t + z - \sqrt{(n-1)^2t^2 - 2(n+1)tz + z^2}}{2tz}$$

$$G_{c^t} = \frac{(2t-1)z + (1-n)t - \sqrt{(n-1)^2t^2 - 2(n+1)tz + z^2}}{2z((t-1)z + n + t(1-n))}$$

We can recover the distribution of c^t with respect to ϕ from the G -transform. It turns out that it has atoms at $\frac{n+t(1-n)}{1-t} = n + \frac{1}{\alpha}$ provided that $t \notin \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}}\right]$.

Actually an eigenvector for the eigenvalue $n + \frac{1}{\alpha}$ is given by

$$\zeta = \sqrt{t}\Omega + \sum_{k \geq 1} \frac{1}{(n\alpha)^k} \sum_{\substack{|\underline{i}|=2k \\ i_{2j+1}=i_{2j+2}}} e_{\underline{i}}$$

Second proof : We use the Khinchine inequalities for free products see [9, 7].

Consider the operator

$$T_k = \sum_{\substack{p \geq 1 \\ k_1 + \dots + k_p = 2k \\ k_{\text{even}} \\ i_1 \in \{1, \dots, n\} \\ i_1 \neq i_2 \neq \dots \neq i_p}} u_{k_1}(s_{i_1}^t) u_{k_2}(s_{i_2}^t) \dots u_{k_p}(s_{i_p}^t)$$

In the Calkin algebra, we can identify

$$\rho(T_k) = \sum_{\substack{p \geq 1 \\ k_1 + \dots + k_p = 2k \\ k_i \text{ even} \\ i_1 \in \{1, 2\} \\ i_1 \neq i_2 \neq \dots \neq i_p}} u_{k_1}(\sqrt{t}s_{i_1}^1)u_{k_2}(\sqrt{t}s_{i_2}^1)\dots u_{k_p}(\sqrt{t}s_{i_p}^1)$$

Thus, we have

$$\rho(T_k)\Omega = \sum_{\substack{|\underline{i}|=2k \\ i_{2j+1}=i_{2j+2}}} e_{\underline{i}}$$

And $\|\rho(T_k)\Omega\|_2 = n^{k/2}$.

As $\rho(T_k)\Omega$ is homogeneous of degree $2k$, from the Khinchine inequalities

$$\|\rho(T_k)\| \leq (2k+1)\|\rho(T_k)\Omega\|_2$$

But expanding the polynomials using conditional freeness, it comes that $\phi(T_k) = n^k \alpha^k$.

So if $n|\alpha| > \sqrt{n}$ then ρ is not isometric and hence there is a compact operator in $C_{t,n}$. \blacksquare

Remark 3.12 Let

$$f(x) = \frac{1}{2x\pi} \frac{\sqrt{(x-t(1-\sqrt{n})^2)(t(1+\sqrt{n})^2-x)}}{(t-1)x+n+t(1-n)} \mathbf{1}_{[t(1-\sqrt{n})^2, t(1+\sqrt{n})^2]}$$

The distribution of c^t with respect to ϕ is

$$f(x)dx \quad \text{if } t \in \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}} \right] \\ f(x)dx + \frac{(n-1)\left(t - \frac{n}{n+\sqrt{n}}\right)\left(t - \frac{n}{n-\sqrt{n}}\right)}{(n(1-t)^2 + t(1-t))} \delta_{n+\frac{1}{\alpha}} \quad \text{if } t \notin \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}} \right]$$

Lemma 3.13 We have $\ker(c^t - (n + \frac{1}{\alpha})) = \mathbb{C}\zeta$.

Proof : We will prove it in several steps. We already know that ζ is one eigenvector for c^t . By valuation of a vector in \mathcal{F}_t , we mean the index of its first non zero component according to the natural filtration of \mathcal{F}_t . So ζ has valuation 0. In the following, we let ξ be one eigenvector (if it exists !) not in $\mathbb{C}\zeta$. We can assume that the valuation of ξ is bigger than 1 and that ξ is real.

From the computation of the G -transforms above, the spectrum of tc^1 is exactly the interval $[t(1-\sqrt{n})^2, t(1+\sqrt{n})^2]$ (as τ is faithful).

First step : The valuation of ξ is 1.

First notice that on tensors of length bigger than 3, c^t acts exactly as tc^1 . Since $\|tc^1\| < n + \frac{1}{\alpha}$, there is no eigenvector with valuation bigger than 3.

Now we focus on the valuation 2. On $\mathcal{H} \otimes \mathcal{H}$. We consider the basis given by $f_1 = e_{11} + \dots + e_{nn}$, $f_2 = e_{11} - e_{22}, \dots, f_n = e_{11} - e_{nn}$, and the vectors $f_{ij} = e_{ij}$ with $i \neq j$.

First the component of ξ on f_1 is 0. Indeed, it is clear that $\langle c^t f_1, \Omega \rangle = n$ and for other basis vectors f (and vectors of valuation bigger than 3), we have $\langle c^t f, \Omega \rangle = 0$. So if ξ has valuation 2, necessarily

$$0 = (n + \frac{1}{\alpha})\langle \xi, \Omega \rangle = \langle c^t \xi, \Omega \rangle = \sqrt{n} \langle \xi, f_1 \rangle$$

On the other hand, as above, on vectors of valuation 2 with no component on f_1 , c^t acts as tc^1 . So there is no such vectors.

Consequently, $\dim \ker c^t - (n + \frac{1}{\alpha}) \leq n + 1$. Since c^t is invariant by conjugation with unitaries coming from the first quantization. If ξ has valuation 1, we can assume that the component of degree 1 of ξ is e_1 .

Second step : $\xi \in \text{Span} \{(c^t)^k e_1 ; k \geq 1\}$

Let f be a continuous function on \mathbb{R} vanishing on the spectrum of tc^1 strictly smaller than 1 except that $f(n + \frac{1}{\alpha}) = 1$. As $\rho(c^t) = tc^1$, it follows that $f(c^t)$ is self-adjoint compact. Modifying f , we can assume that $f(c^t)$ is exactly the projection onto $\ker c^t - (n + \frac{1}{\alpha})$.

Then, $f(c^t)e_1$ is non zero as

$$\langle f(c^t)e_1, \xi \rangle = \langle e_1, f(c^t)\xi \rangle = \langle e_1, \xi \rangle = 1$$

So $f(c^t)e_1$ is one eigenvector. It is collinear with ξ for its component of degree 1 is collinear to e_1 and is non vanishing as there is no eigenvector of valuation greater than 2.

Third step : ξ doesn't exist.

It is clear by induction that $\text{Span}\{(c^t)^k e_1 ; k \geq 0\} = \text{Span}\{f_k, k \geq 0\}$ with

$$f_k = \sum_{\substack{|\underline{i}|=2k \\ i_{2j+1}=i_{2j+2}}} e_{\underline{i}1}$$

We let $\xi = \sum_{k \geq 0} x_k f_k$. we have

$$\begin{aligned} c^t f_k &= nt f_{k-1} + (n+1) t f_k + f_{k+1} \quad k \geq 1 \\ c^t f_0 &= (nt+1) f_0 + t f_1 \end{aligned}$$

Consequently the sequence x_i has to satisfy the recursion formula :

$$\begin{aligned} (n + \frac{1}{\alpha}) x_0 &= (nt+1) x_0 + n t x_1 \\ (n + \frac{1}{\alpha}) x_k &= n t x_{k+1} + (n+1) t x_k + t x_{k-1} \quad k \geq 1 \end{aligned}$$

Hence, $x_k = a(n\alpha)^{-k} + b\alpha^k$, with $b \neq 0$ as soon as $n \neq 1$.

But then with this values the series $\sum x_k f_k$ is not convergent ! So ξ doesn't exists ■

Theorem 3.14 For $n \geq 2$ and $t \notin \left[\frac{n}{n+\sqrt{n}}, \frac{n}{n-\sqrt{n}} \right]$, as von Neumann algebras, we have

$$\Gamma_{t,n} = \mathbb{B}(\ell_2) \oplus \Gamma_{1,n}$$

Moreover the state ϕ is not faithful on $\Gamma_{t,n}$.

Proof : Let $1 - q$ be the central support of the representation ρ . As above, we know that $q \neq 0$. We have to show that $q\Gamma_{t,n} = \mathbb{B}(\ell_2)$.

Since Ω is cyclic for $\Gamma_{t,n}$, $q\Omega$ is cyclic in $q\mathcal{F}_t$ for $q\Gamma_{t,n}$ (hence non zero). Let p be the projection onto $\mathbb{C}\zeta$. We have that for $x \in \Gamma_{t,n}$,

$$0 = \tau(\rho(qx)) = \langle qx\Omega, \eta \rangle = \alpha \langle x\Omega, (c^t - (n + \frac{1}{\alpha}))q\Omega \rangle$$

So $(c^t - (n + \frac{1}{\alpha}))q\Omega = 0$, and $q\Omega = \lambda\zeta$ for some $\lambda \neq 0$. So $p \leq q$, and $q\Gamma_{t,n}$ contains a one dimensional projection on a cyclic vector so is isomorphic to $\mathbb{B}(\ell_2)$ (as $\Gamma_{t,n}\zeta$ is infinite dimensional). The state can not be faithful because of the $\mathbb{B}(\ell_2)$ summand. ■

3.4 Case $n/(n + \sqrt{n}) < t < n/(n - \sqrt{n})$

At the algebraic level, we have already seen that for $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$, with $\underline{i} = i_1^{\alpha_1} \dots i_l^{\alpha_l}$:

$$u_{\alpha_1}(s_{i_1}^t) \dots u_{\alpha_{l-1}}(s_{i_{l-1}}^t) v_{\alpha_l}(s_{i_l}^t) \Omega = e_{\underline{i}}$$

We define an antilinear map S on $\text{Span } e_{\underline{i}}$ by

$$S(e_{\underline{i}}) = v_{\alpha_l}(s_{i_l}^t) u_{\alpha_{l-1}}(s_{i_{l-1}}^t) \dots u_{\alpha_1}(s_{i_1}^t) \Omega$$

Lemma 3.15 *The map S satisfies that for any i_1, \dots, i_l (not necessarily with $i_1 \neq \dots \neq i_l$) and any polynomials P_1, \dots, P_l :*

$$S(P_1(s_{i_1}^t) \dots P_l(s_{i_l}^t) \Omega) = \phi P_l(s_{i_l}^t) \dots \phi P_1(s_{i_1}^t) \Omega$$

Proof : Clear by induction on n . ■

Lemma 3.16 *For $\frac{n}{n+\sqrt{n}} < t < \frac{n}{n-\sqrt{n}}$, S extends to a bounded operator.*

Proof : We decompose S . Let $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$ and define antilinear operators by :

$$\begin{aligned} A(e_{\underline{i}}) &= u_{\alpha_l}(s_{i_l}^t) u_{\alpha_{l-1}}(s_{i_{l-1}}^t) \dots u_{\alpha_1}(s_{i_1}^t) \Omega \\ B(e_{\underline{i}}) &= u_{\alpha_{l-2}}(s_{i_l}^t) u_{\alpha_{l-1}}(s_{i_{l-1}}^t) \dots u_{\alpha_1}(s_{i_1}^t) \Omega \end{aligned}$$

with the convention that $u_k = 0$ if $k < 0$. Then $S = \sqrt{t}(A - \alpha B)$.

From the relations between v and u , it follows that A sends tensors of length l to a sum of tensors of length $l, l-2, \dots$. Fix $k \geq 0$, and denote by A_k the component of A that sends tensors of length l to tensors of length $l-2k$. Let J be the antiunitary of \mathcal{F}_t that reverses the order of tensors. Put $C = \sum_{s=1}^n l_1(e_s)^2 J$, then it is not hard to see that for $|\underline{i}| - |\underline{j}| = 2k$:

$$\langle A_k e_{\underline{i}}, e_{\underline{j}} \rangle = \alpha^k f(\underline{i}, \underline{j}) \langle C^k e_{\underline{i}}, e_{\underline{j}} \rangle$$

where $f(\underline{i}, \underline{j})$ can take the value 1 or $\frac{1}{\sqrt{t}}$.

Since the coefficients of C are all positive, we get that

$$\|A_k\| \leq \frac{|\alpha|^k}{\sqrt{t}} \|C^k\|$$

And as $C^*C = n\text{Id}$, we get that

$$\|A\| \leq \frac{1}{\sqrt{t}} \sum_{k=0}^{\infty} (\sqrt{n}|\alpha|)^k$$

which is convergent provided that $\frac{n}{n+\sqrt{n}} < t < \frac{n}{n-\sqrt{n}}$.

The same kind of arguments shows that B is bounded. ■

Now, we assume that $\frac{n}{n+\sqrt{n}} < t < \frac{n}{n-\sqrt{n}}$.

Lemma 3.17 *One has that for any $i_1 \neq \dots \neq i_l$ and $k \leq n$ and any polynomials P_1, \dots, P_l :*

$$[SP_1(s_{i_1}^t) \dots P_l(s_{i_l}^t) S, s_k^t] = 0$$

Proof : Just a writing game, and the boundedness of S ■

Corollary 3.18 *For $\frac{n}{n+\sqrt{n}} < t < \frac{n}{n-\sqrt{n}}$, the state ϕ is faithful on $\Gamma_{t,n}$.*

Proof : Ω is cyclic for $\Gamma'_{t,n}$ as by the previous lemma $S\Gamma_{t,n}S \subset \Gamma'_{t,n}$ and S is invertible (because $S^2 = \text{Id}$). ■

Corollary 3.19 *For $\frac{n}{n+\sqrt{n}} < t < \frac{n}{n-\sqrt{n}}$, $\Gamma_{t,n}$ does not contain any compact operator. Moreover the C^* -algebras $C_{t,n}$ and $C_{1,n}$ are isomorphic.*

Proof : If $\Gamma_{t,n}$ contains a compact operator, then as above there is a direct summand of $\Gamma_{t,n}$ isomorphic to $\mathbb{B}(\ell_2(I))$. But as Ω is separating, I must consist of one point, which is impossible as s_1^t as no eigenvector.

The second assertion follows from the first one since the Calkin map is then an isomorphism in both case. ■

Theorem 3.20 *For $\frac{n}{n+\sqrt{n}} < t < \frac{n}{n-\sqrt{n}}$, the von Neumann algebra $\Gamma_{t,n}$ is isomorphic to $\Gamma_{1,n}$.*

Proof : One just needs to show that the map ρ is faithful. To do so it suffices to show that ψ is faithful.

Let $b = S(1 - \sqrt{t}\alpha \sum_{i=1}^n v_2(s_i^t))S \in \Gamma'_{t,n}$. We have for $x \in \Gamma_{t,n}$

$$\psi(x) = \langle b^* x \Omega, \Omega \rangle.$$

Applying it to $x = y^*y$ and as Ω is cyclic it follows that b is positive. Say $b = a^2$, then

$$\psi(x^*x) = \|xa\Omega\|^2$$

Moreover the distribution of b for ϕ is absolutely continuous with respect to the Lebesgue measure (see Remark 3.12), it follows that there is a net of elements (c_i) in the von Neumann generated by b so that $c_i a \Omega \rightarrow \Omega$. Now, if for $x \in \Gamma_{t,n}$, $\psi(x^*x) = 0$ then

$$0 = \lim c_i x a \Omega = \lim x c_i a \Omega = x \Omega.$$

So $x = 0$ as Ω is cyclic. ■

Remark 3.21 The Tomita-Takesaki operator S is bounded on $\Gamma_{t,n}$.

Remark 3.22 It is possible to reverse all the arguments above. To do so, define a normal linear form $\tilde{\phi}$ on $\Gamma_{1,n}$ by

$$\tilde{\phi}(x) = \langle x \Omega, \sum_{k \geq 0} \alpha^k \sum_{\substack{|i|=2k \\ i_{2j+1}=i_{2j+2}}} e_i \rangle$$

It is well defined because the vector on the right side makes sense as $\sqrt{n}|\alpha| < 1$

It is clear that on the set of polynomials in s_i^1 , it coincides with ϕ . By continuity, we get that on $\Gamma_{t,n}$, we have

$$\phi = \tilde{\phi} \rho$$

Then the GNS construction of $\Gamma_{1,n}$ for $\tilde{\phi}$ gives ρ^{-1} .

3.5 Case $t = n/(n \pm \sqrt{n})$

In this situation, we will adopt the strategy of the last remark.

Let \mathcal{P} be the set of noncommutative polynomials in the letters X_1, \dots, X_n . It has a natural $*$ -algebra structure.

For $0 < r < 1$, we define a linear functional ϕ_r on \mathcal{P} : for $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$

$$\phi_r(u_{\alpha_1}(X_{i_1}) \dots u_{\alpha_l}(X_{i_l})) = \begin{cases} (r^2 \alpha)^{\sum \alpha_i / 2} & \text{if all } \alpha_i \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$

Formally, we can identify \mathcal{P} with the $*$ -algebra generated by $\sqrt{t}s_i^1$ in $C_{1,n}$ (say $\pi(X_i) = \sqrt{t}s_i^1$) and to $*$ -algebra generated by s_i^t in $C_{t,n}$ (say $\sigma(X_1) = s_i^t$).

Let T_r be the second quantization associated to $r\text{Id}$ on $\Gamma_{1,n}$, it is unital completely positive and let $\pi(\mathcal{P})$ invariant.

The functional ϕ_r is made so that

$$\phi_r(P) = \phi(\sigma(\pi^{-1}(T_r(\pi(P)))))$$

Indeed, for $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$

$$\begin{aligned} \pi^{-1}(T_r(\pi(u_{\alpha_1}(X_{i_1}) \dots u_{\alpha_l}(X_{i_l})))) &= \pi^{-1}(T_r(u_{\alpha_1}(\sqrt{t}s_{i_1}^1) \dots u_{\alpha_l}(\sqrt{t}s_{i_l}^1))) \\ &= \pi^{-1}(r^{\sum \alpha_i} u_{\alpha_1}(\sqrt{t}s_{i_1}^1) \dots u_{\alpha_l}(\sqrt{t}s_{i_l}^1)) \\ &= r^{\sum \alpha_i} u_{\alpha_1}(X_{i_1}) \dots u_{\alpha_l}(X_{i_l}) \end{aligned}$$

and by conditional freeness :

$$\phi(\sigma(u_{\alpha_1}(X_{i_1}) \dots u_{\alpha_l}(X_{i_l}))) = \phi(u_{\alpha_1}(s_{i_1}^t) \dots u_{\alpha_l}(s_{i_l}^t)) = \begin{cases} \alpha^{\sum \alpha_i / 2} & \text{if all } \alpha_i \text{ are even} \\ 0 & \text{otherwise} \end{cases}$$

Lemma 3.23 *The functional ϕ_r is positive on \mathcal{P} .*

Proof : The $*$ -algebra A_i generated by X_i in \mathcal{P} has two particular functionals on it. The first one is the restriction of ϕ_r (still denoted by ϕ_r) and the second one is $\psi\pi$. From the definition of ϕ_r the algebra $(A_i, \phi_r, \psi\pi)$ are conditionally free. Hence the result will follow from [4] Theorem 2.2 provided that we can show the positivity of ϕ_r and $\psi\pi$ on each A_i . For $\psi\pi$ this is obvious as ψ is a state. For ϕ_r , it suffices to notice that the distribution of X_i for ϕ_r the distribution of rs_i^t for ϕ , so comes from a probability measure. \blacksquare

Corollary 3.24 *The functional $\phi_r\pi^{-1}$ extends to a normal state on $\Gamma_{1,n}$ (denoted by ϕ_r).*

Proof : By the preceding Lemma, it is positive on $\pi(\mathcal{P})$ which is weak-* dense in $\Gamma_{1,n}$. The result follows from the representation

$$\phi_r(\pi^{-1}(x)) = \langle x\Omega, \sum_{k \geq 0} (r^2\alpha)^k \sum_{\substack{|\underline{i}|=2k \\ i_{2j+1}=i_{2j+2}}} e_{\underline{i}} \rangle$$

It is well defined because the vector on the right side makes sense as $r^2\sqrt{n}|\alpha| = r^2 < 1$. \blacksquare

Theorem 3.25 *The C^* -algebras $C_{n/(n \pm \sqrt{n}), n}$ and $C_{1,n}$ are isomorphic and the state ϕ is faithful on $C_{n/(n \pm \sqrt{n}), n}$.*

Proof : There is a state on $C_{1,n}$ defined by $\tilde{\phi} = \lim_{r \rightarrow 1, \mathfrak{U}} \phi_r$, where the limit is taken along some non trivial ultrafilter. It is clear from the formulas on \mathcal{P} , that for $x \in \mathbb{C}_{t,n}$, we have

$$\phi(x) = \tilde{\phi}(\rho(x)).$$

We consider the GNS construction of $\tilde{\phi}$ for $C_{1,n}$. The underlining Hilbert space is exactly $\mathcal{F}_{n/n \pm \sqrt{n}}$ from the above identities, so this GNS construction is the inverse of the Calkin map ρ .

Assume $x \in C_{n/(n \pm \sqrt{n}), n}^+$ is such that $\phi(x^2) = 0$, then $x\Omega = 0$. It follows that $\psi(\rho(x)) = \langle x\Omega, \eta \rangle = 0$. So $\rho(x) = 0$ as ψ is faithful on $\Gamma_{1,n}$, so $x = 0$. \blacksquare

The analogue result for the von Neumann algebras is a consequence of the next section. In particular they show that the map $\tilde{\phi}$ is actually normal.

3.6 Case $n = \infty$

This case was already treated in [16] and was a starting point to this work.

If $t \neq 1$ then it can be checked that

$$\frac{1}{k} \sum_{i=1}^k (s_i^t)^2 \xrightarrow{k \rightarrow \infty} (1-t)P + t\text{Id}$$

in the weak-* topology, where P is the orthogonal projection onto Ω . As Ω is cyclic, Proposition 3.1 ensures that

$$\Gamma_{t,\infty} = \mathbb{B}(\mathcal{F}_t)$$

4 Generalizations

In this section, we explain how to extend the previous results.

Let I_i be bounded measurable subsets of \mathbb{R} , $A_i = L_\infty(I_i, \mu_i)$ be n commutative von Neumann algebras. The integration with respect to μ_i will be called ϕ_i and s_i is the identity function on I_i . Assume that we are given ψ_i , distinguished normal states on A_i . As a consequence, ψ_i has a density f_i with respect to ϕ_i , we put $\tilde{f}_i = f_i - 1$. We will denote by v_k^i and u_k^i the orthogonal polynomials for ϕ_i and ψ_i .

We consider the algebraic conditional free product $\tilde{A} = *_{i=1}^n (A_i, \phi_i, \psi_i)$. The state ϕ is the conditional free product state and ψ is the free product of the ψ_i . Then we can talk about the von Neumann reduced conditional free product, corresponding to GNS construction of ϕ , we denote it by A

$$A \subset \mathbb{B}(\mathcal{F}) \quad \phi = \langle \cdot, \Omega, \Omega \rangle$$

\mathcal{F} is the conditional Fock space and Ω the vacuum state, we refer to [10] Section 6 for more details about this construction. One of the main trouble comes from the fact that in general the state ϕ is not faithful on A (for t -gaussians for instance). Nevertheless, the injections $i_i : A_i \rightarrow A$ are normal and isometric and preserve the states (i.e. $i_i^*(\phi) = \phi_i$, so we will drop the indexes) (this corresponds to Proposition 2.1). Unfortunately, it seems that in general there is no conditional expectation from A to A_i which is state preserving.

In this situation, the Lemmas 3.3 and 3.4 remain true. For $i_1 \neq \dots \neq i_l$ and $\alpha_j \geq 1$, let $\underline{i} = i_1^{\alpha_1} \dots i_l^{\alpha_l}$ and define :

$$e_{\underline{i}} = u_{\alpha_1}^{i_1}(s_{i_1}) \dots u_{\alpha_{l-1}}^{i_{l-1}}(s_{i_{l-1}}) v_{\alpha_l}^{i_l}(s_{i_l}) \Omega$$

Lemma 4.1 *The family $(e_{\underline{i}})_{\underline{i} \in \{1, \dots, n\}^{<\infty}}$ is an orthonormal basis of \mathcal{F} .*

If the densities f_i are bounded (or in $L^2(\mu_i)$), the state ψ on the algebraic free product extends to a normal state of A .

Proof : To prove the first assertion we will need to evaluate expressions of the form $\phi(ab_1 \dots b_l d)$, where $a \in A_{i_0}$, $b_k \in A_{i_k}$, $\psi_{i_k}(b_k) = 0$ and $d \in A_{i_{l+1}}$. With $b = b_1 \dots b_l$ and $x = \phi_{i_1}(b_1) \dots \phi_{i_l}(b_l)$

$$\begin{aligned} \phi(abd) &= \phi((a - \psi_{i_0}(a))b(d - \psi_{i_{l+1}}(d))) + \psi_{i_0}(a)\phi(b(d - \psi_{i_{l+1}}(d))) + \\ &\quad \phi((a - \psi_{i_0}(a))b)\psi_{i_{l+1}}(d) + \psi_{i_0}(a)\phi(b)\psi_{i_{l+1}}(d) \\ &= (\phi(a) - \psi_{i_0}(a))x(\phi(d) - \psi_{i_{l+1}}(d))\psi(a)x(\phi(d) - \psi_{i_{l+1}}(d)) + \\ &\quad + (\phi(a) - \psi_{i_0}(a))x\psi_{i_{l+1}}(d) + \psi_{i_0}(a)x\psi_{i_{l+1}}(d) \\ &= \phi(a)x\phi(d) \\ &= \phi(a)\phi(b)\phi(d) \end{aligned}$$

For $\underline{i} = i_1^{\alpha_1} \dots i_l^{\alpha_l}$ and $\underline{j} = j_1^{\beta_1} \dots j_m^{\beta_m}$, we have

$$\langle e_{\underline{i}}, e_{\underline{j}} \rangle = \phi\left(v_{\beta_m}^{j_m}(s_{j_m}) u_{\beta_{m-1}}^{j_{m-1}}(s_{j_{m-1}}) \dots u_{\beta_1}^{j_1}(s_{j_1}) u_{\alpha_1}^{i_1}(s_{i_1}) \dots u_{\alpha_{l-1}}^{i_{l-1}}(s_{i_{l-1}}) v_{\alpha_l}^{i_l}(s_{i_l})\right)$$

So if $i_1 \neq j_1$ or $\alpha_1 \neq \beta_1$ then as $\phi(v_{\alpha_l}^{i_l}(s_{i_l})) = 0$, the previous identity ensures that this scalar product is 0. In the remaining case $i_1 = j_1$ and $\alpha_1 = \beta_1$, so $\psi_{i_1}(u_{\beta_1}^{j_1}(s_{j_1}) u_{\alpha_1}^{i_1}(s_{i_1})) = 1$ and the above identity gives

$$\langle e_{\underline{i}}, e_{\underline{j}} \rangle = \langle e_{i_2^{\alpha_2} \dots i_l^{\alpha_l}}, e_{j_2^{\beta_2} \dots j_l^{\beta_l}} \rangle$$

and an induction completes the proof as the v_n^i are orthonormal.

If the f_i are bounded, then

$$c = 1 + \sum_{i=1}^n \hat{f}_i(s_i) \in A$$

and for all $x \in \tilde{A}$,

$$\phi(cx) = \psi(x)$$

By linearity and freeness, it suffices to check it for $x = u_{\alpha_1}^{i_1}(s_{i_1}) \dots u_{\alpha_{l-1}}^{i_{l-1}}(s_{i_{l-1}}) u_{\alpha_l}^{i_l}(s_{i_l})$ with $l \geq 1$,

$\alpha_k > 0$ and $i_1 \neq \dots \neq i_l$ as $\phi(c) = 1$. We have

$$\begin{aligned}\phi(cx) &= \phi\left(1 + \sum_{i=1}^n \hat{f}_i(s_i)\right)u_{\alpha_1}^{i_1}(s_{i_1}) \dots u_{\alpha_{l-1}}^{i_{l-1}}(s_{i_{l-1}})u_{\alpha_l}^{i_l}(s_{i_l})\right) \\ &= \phi\left(f_{i_1}(s_{i_1})u_{\alpha_1}^{i_1}(s_{i_1}) \dots u_{\alpha_{l-1}}^{i_{l-1}}(s_{i_{l-1}})u_{\alpha_l}^{i_l}(s_{i_l})\right) \\ &= \phi(f_{i_1}(s_{i_1})u_{\alpha_1}^{i_1}(s_{i_1}))\phi(u_{\alpha_2}^{i_2}(s_{i_2})) \dots \phi(u_{\alpha_l}^{i_l}(s_{i_l})) \\ &= \psi(u_{\alpha_1}^{i_1}(s_{i_1}))\phi(u_{\alpha_2}^{i_2}(s_{i_2})) \dots \phi(u_{\alpha_l}^{i_l}(s_{i_l})) \\ &= 0\end{aligned}$$

This gives the normality and the extension of ψ as $c \in A$.

If the f_i are only in L^2 , then we can consider

$$\eta = \left(1 + \sum_{i=1}^n \hat{f}_i(s_i)\right).\Omega$$

and simply replace $\phi(cx)$ by $\langle x\Omega, \eta \rangle$ in the above proof. ■

Remark 4.2 We need the assumption on the densities because of the lack of conditional expectation. So we can not define c correctly when f_i are only in L^1 . It is very likely that in concrete situations one can find an appropriate proof of this result.

Corollary 4.3 If $f_i \in L^2(\mu_i)$, the conditional free product $*_{i=1}^n(A_i, \phi_i, \psi_i)$ has a direct summand isomorphic to the free product $*_{i=1}^n(A_i, \psi_i)$.

Remark 4.4 In order to have isometric copies of A_i in the free product $*_{i=1}^n(A_i, \psi_i)$, one needs to assume that the GNS construction of A_i for ψ_i to be faithful. So it boils down to assume that ϕ_i is absolutely continuous with respect to ψ_i , this corresponds to the case $t \geq \frac{1}{2}$ for t -gaussians.

Remark 4.5 The basis e_i is natural in the sense that on it, the generators acts like gaussian operators (creation plus annihilation). In general, one does not recover interacting Fock spaces.

Theorem 4.6 Assume that the densities f_i are bounded and that the distribution of c with respect to ϕ does not have atom at 0. Then as von Neumann algebras

$$*_{i=1}^n(A_i, \phi_i, \psi_i) = *_{i=1}^n(A_i, \psi_i)$$

If moreover ψ_i is faithful on A_i for $i = 1, \dots, n$ then the state ϕ is faithful.

Proof : We denote by ρ the representation from the conditional free product to the free product.

The GNS representation for ϕ restricted to the von Neumann generated by c is included in the reduced conditional free product. Then the assumption on c ensures that one can find b_n (self-adjoint) in the C^* -algebra generated by c so that $b_n c \Omega \rightarrow \Omega$.

Let x be such that $\rho(x) = 0$. Hence, for any a and d in A ,

$$0 = \lim_{k \rightarrow \infty} \psi(\rho(b_n a x d)) = \lim_{k \rightarrow \infty} \phi(c b_n a x d) = \lim_{k \rightarrow \infty} \langle a x d \Omega, b_n c \Omega \rangle = \langle a x d \Omega, a^* \Omega \rangle = \phi(a x d)$$

So $x = 0$ as Ω is cyclic for A .

If the ψ_i are faithful, then their free product ψ is also faithful by a result of Dykema [8]. Assume x is so that $\phi(x^* x) = 0$, then by the Cauchy-Schwarz inequality, $\phi(cx^* x) = 0$. So one has $\psi(\rho(x^* x)) = 0$ and $\rho(x) = 0$. Consequently $x = 0$ as ρ is one to one. ■

Remark 4.7 If the distribution of c has atom at 0, then the reduced conditional free product is not a factor.

Another approach, to show that the free product and the conditional free products coincide, is to adopt the strategy of the Remark 3.22 or Section 3.5. Unfortunately, it seems that it is not as good as the above Theorem, as one can not use it to get the limit cases $t = \frac{n}{n \pm \sqrt{n}}$.

We let f_i be the natural orthonormal basis in the free Fock space associated to the above distributions. Consider the vector

$$\zeta = \sum_{\underline{i}=i_1^{\alpha_1} \dots i_l^{\alpha_l}} \phi(u_{\alpha_1}(s_{i_1}) \dots u_{\alpha_l}(s_{i_l})) f_{\underline{i}}$$

It exists if and only if

$$\sum_{\underline{i}=i_1^{\alpha_1} \dots i_l^{\alpha_l}} \phi_{i_1}(u_{\alpha_1}(s_{i_1}))^2 \dots \phi_{i_l}(u_{\alpha_l}(s_{i_l}))^2 < \infty.$$

Then, the normal state on $*(A_i, \psi_i)$ defined by

$$\tilde{\phi}(x) = \langle x\Omega, \zeta \rangle$$

coincides with ϕ on \mathcal{P} . So the GNS construction of this normal state gives a representation from the free product to the conditional free product.

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References

- [1] ACCARDI, L., AND BOŽEJKO, M. Interacting Fock spaces and Gaussianization of probability measures. *Infin. Dimens. Anal. Quantum Probab. Relat. Top.* 1, 4 (1998), 663–670.
- [2] BOŽEJKO, M., AND FENDLER, G. A note on certain partial sum operators. Preprint.
- [3] BOŽEJKO, M., KÜMMERER, B., AND SPEICHER, R. q -Gaussian processes: non-commutative and classical aspects. *Comm. Math. Phys.* 185, 1 (1997), 129–154.
- [4] BOŽEJKO, M., LEINERT, M., AND SPEICHER, R. Convolution and limit theorems for conditionally free random variables. *Pacific J. Math.* 175, 2 (1996), 357–388.
- [5] BOŽEJKO, M., AND SPEICHER, R. Completely positive maps on Coxeter groups, deformed commutation relations, and operator spaces. *Math. Ann.* 300, 1 (1994), 97–120.
- [6] BOŽEJKO, M., AND WYSOCZAŃSKI, J. Remarks on t -transformations of measures and convolutions. *Ann. Inst. H. Poincaré Probab. Statist.* 37, 6 (2001), 737–761.
- [7] BUCHHOLZ, A. L_∞ -Khintchine-Bonami inequality in free probability. In *Quantum probability (Gdańsk, 1997)*, vol. 43 of *Banach Center Publ.* Polish Acad. Sci., Warsaw, 1998, pp. 105–109.
- [8] DYKEMA, K. J. Faithfulness of free product states. *J. Funct. Anal.* 154, 2 (1998), 323–329.
- [9] HAAGERUP, U. An example of a nonnuclear C^* -algebra, which has the metric approximation property. *Invent. Math.* 50, 3 (1978/79), 279–293.
- [10] MŁOTKOWSKI, W. Operator-valued version of conditionally free product. *Studia Math.* 153, 1 (2002), 13–30.
- [11] NOU, A. Non injectivity of the q -deformed von Neumann algebra. *Math. Ann.* 330, 1 (2004), 17–38.
- [12] RICARD, É. Factoriality of q -gaussian von neumann algebras. *Comm. Math. Phys.* 257, 2 (2005), 659–665.
- [13] SHLYAKHTENKO, D. Some estimates for non-microstates free entropy dimension with applications to q -semicircular families. *Int. Math. Res. Not.*, 51 (2004), 2757–2772.
- [14] VOICULESCU, D. V., DYKEMA, K. J., AND NICÀ, A. *Free random variables*, vol. 1 of *CRM Monograph Series*. American Mathematical Society, Providence, RI, 1992. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups.

- [15] WOJAKOWSKI, Ł. *Probabilistyka interpolujaca pomiedzy wolna boolowska.* PhD thesis, Wrocław University, 2004.
- [16] WYSOCZAŃSKI, J. The von Neumann algebra associated with t -free non-commutative gaussian random variables. notes.